

# PROJECTIONS AND DIMENSION CONSERVATION FOR RANDOM SELF-SIMILAR MEASURES AND SETS

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**ABSTRACT.** This paper studies the geometric properties of random multiplicative cascade measures on self-similar sets. We prove that these classes of measures are exact-dimensional, generalizing Feng and Hu's result on self-similar measures. One consequence is that they obey the dimension conservation property introduced by Furstenberg, extending his conclusions for 'homogeneous' self-similar measures to a much larger class of measures. We also generalize Hochman and Shmerkin's theorem on projections of self-similar measures to random multiplicative cascade measures on self-similar sets without any separation conditions. We give some applications to projections, dimension conservation and distance sets of self-similar sets and to fractal percolation on self-similar sets.

## 1. INTRODUCTION

Relating the Hausdorff dimension of a set  $K \subseteq \mathbb{R}^d$  to those of its sections and projections has a long history. The most basic result is that if  $K$  is Borel or analytic, then, for almost all orthogonal projections  $\pi$  from  $\mathbb{R}^d$  to its  $k$ -dimensional subspaces (with respect to the natural invariant measure on projections)

$$\dim_H \pi K = \min(k, \dim_H K),$$

where  $\dim_H$  denotes Hausdorff dimension. This was proved in the plane by Marstrand [20] and extended to general  $d$  and  $k$  by Mattila [22]. Kaufman [16] introduced the potential theoretic method which is now commonly used in addressing such problems. These papers [20, 22] also consider the dimensions of sections or fibres of sets and show that for almost all  $k$ -dimensional subspaces  $V$ , with  $\pi$  denoting orthogonal projection onto  $V$ , the sections  $\pi^{-1}x \cap K$  have dimensions satisfying  $\dim_H(\pi^{-1}x \cap K) \leq \max(0, \dim_H K - k)$  for Lebesgue almost all  $x \in V$  but with equality for a set of  $x \in V$  of positive  $k$ -dimensional measure (we take  $\dim_H \emptyset = -\infty$ ). This is complemented by the fact [21] that for *all*  $k$ -dimensional subspaces  $V$ ,

$$(1.1) \quad \Delta + \dim_H \{y \in V : \dim_H(K \cap \pi^{-1}y) \geq \Delta\} \leq \dim_H K$$

for all  $0 \leq \Delta \leq d - k$ . A good exposition of this material may be found in [23].

Such results have been extended beyond recognition, for example to families of generalized projections [27], to obtain estimates on the size of 'exceptional' projections for which the conclusions fail [27], and to packing dimensions [9]. Nevertheless, almost all of this work concerns sections and projections of general (Borel or analytic) sets  $K$  for which the possibilities of exceptional projections can never be excluded. In a few specific cases there has been a careful analysis of how the dimensions of projections or intersections can differ significantly in different directions, for

example for the 1-dimensional Sierpinski triangle [17] and for the Sierpinski carpet [19].

Recently, attention has turned to projections and slices of specific classes of sets, in particular self-similar sets. Powerful techniques from ergodic theory have been harnessed to obtain results that apply to *all*, rather than almost all projections or sections of self-similar sets. Notably, Hochman and Shmerkin [13] have shown that all projections of self-similar sets satisfying strong separation have Hausdorff dimension  $\min(k, \dim_H K)$  provided that the rotational components of the transformations defining the self-similar sets generate a group that is dense in the full orthogonal group. Also Furstenberg [11] has shown that self-similar sets that are ‘homogeneous’, including self-similar sets where the similarities have no rotational component, are ‘dimension conserving’, that is there is a value of  $\Delta$  such that the opposite inequality to (1.1) holds.

Historically, most work has concerned projections and sections of sets, but many of the ideas apply more generally to projections and sections of measures. Indeed, properties of sets have frequently been obtained by proving results for measures which are then applied to suitable measures supported by the sets. Here we study the dimensions of projections and fibres of random multiplicative cascade measures supported on self-similar subsets of  $\mathbb{R}^d$  [15, 2]. We establish results for projections in all directions and dimension conservation properties in this context. In the final two sections of the paper we specialise to self-similar sets and percolation on general self-similar sets to improve existing results and obtain new ones.

Let

$$(1.2) \quad \mathcal{I} = \{f_i = r_i O_i \cdot + t_i\}_{i=1}^m$$

be an iterated function system (IFS) of contractions on  $\mathbb{R}^d$ , where  $f_i$  has contraction ratio  $r_i$ , orthonormal rotation  $O_i$  and translation  $t_i$ . Such an IFS has an *attractor*  $K$ , namely the unique non-empty compact set  $K$  such that

$$(1.3) \quad K = \bigcup_{i=1}^m f_i(K).$$

Let  $\Pi_{d,k}$  be the family of orthogonal projections from  $\mathbb{R}^d$  to its  $k$ -dimensional subspaces. Hochman and Shmerkin [13] proved that, if  $\mathcal{I}$  satisfies the strong separation condition (SSC), that is if the union in (1.3) is disjoint, and the group generated by  $O_i$ ,  $i = 1, \dots, m$  is dense in  $SO(d, \mathbb{R})$ , then for any measure  $\mu$  that is self-similar with respect to the iterated function system  $\mathcal{I}$ ,

$$(1.4) \quad \dim_H \pi \mu = \min(k, \dim_H \mu) \text{ for all } \pi \in \Pi_{d,k}.$$

By approximating  $C^1$  mappings by orthogonal projections, (1.4) may be generalized to

$$(1.5) \quad \dim_H g\mu = \min(k, \dim_H \mu)$$

for any  $C^1$  mapping  $g : K \mapsto \mathbb{R}^k$  without singular points, that is with derivative matrix non-singular. In [25] Orponen uses this fact to solve the distance set problem for self-similar sets in  $\mathbb{R}^2$ , namely that if a self-similar set has Hausdorff dimension greater than 1 then its distance set has dimension 1.

The key ingredient in Hochman and Shmerkin’s proof is the CP-chain technique introduced by Furstenberg [11] that provides a measure-valued ergodic sequence under the product probability measure  $\mu \times \mathbb{P}$ , where  $\mathbb{P}$  is an auxiliary probability

measure on a suitable probability space. The ergodicity ensures that a family of expressions converge simultaneously, so that results such as (1.4) may be extended from ‘for almost every’ to ‘for all’. Such a strategy turns out to be very useful in dealing with this kind of extensions, and in [13] it is also used to prove a conjecture of Furstenberg on the dimensions of sums of invariant sets.

The CP-chain used in [13] for self-similar measures is somehow implicit. It involves scaling and centring the measures and the ergodic decomposition of a stationary measure-valued sequence, see [12]. Due to the simple nature of self-similar measures, one might expect a more direct and explicit argument. Here we use Komolgorov’s zero-one law combined with the group extension theorem to build directly a measure-valued ergodic sequence, which gives (1.4) for random multiplicative cascade measures on self-similar sets but without the need for any separation conditions. This also relies on a generalisation of the exact-dimensional result in [10] for self-similar measures to random cascade measures on self-similar sets. Such a generalisation uses the method in [11, 10] for proving the exact-dimensionality, namely the Maker’s ergodic theory and conditional entropy. It turns out that one can also use this method to deduce the dimension conservation for random cascade measures on self-similar sets for any orthogonal projection without additional assumptions on the IFS.

## 2. PRELIMINARIES

We first review the pertinent background and set up the notation that we will need.

**2.1. Symbolic space.** Symbolic or code space underlies the structure of self-similar sets. Let  $\Lambda = \{1, \dots, m\}$  be the alphabet on  $m \geq 2$  symbols. Denote by  $\Lambda^* = \cup_{n \geq 0} \Lambda^n$  the set of finite words, with the convention that  $\Lambda^0 = \{\emptyset\}$ .

Let  $\Lambda^{\mathbb{N}}$  be the symbolic space of infinite sequences from the alphabet. For  $\underline{i} \in \Lambda^{\mathbb{N}}$  and  $n \geq 0$  let  $\underline{i}|_n \in \Lambda^n$  be the first  $n$  digits of  $\underline{i}$ . For  $i \in \Lambda^n$  let  $[i] = \{\underline{i} \in \Lambda^{\mathbb{N}} : \underline{i}|_n = i\}$  be the cylinder rooted at  $i$ . We may endow  $\Lambda^{\mathbb{N}}$  with the standard metric  $d_\rho$  with respect to a number  $\rho \in (0, 1)$ , that is for  $\underline{i}, \underline{j} \in \Lambda^{\mathbb{N}}$ ,

$$d_\rho(\underline{i}, \underline{j}) = \rho^{\inf\{n \geq 0 : \underline{i}|_n \neq \underline{j}|_n\}}.$$

Then  $(\Lambda^{\mathbb{N}}, d_\rho)$  is a compact metric space. Let  $\mathcal{B}$  be its Borel  $\sigma$ -algebra.

Define the left-shift map  $\sigma$  by  $\sigma(\underline{i}) = (i_{n+1})_{n \geq 1}$  for  $\underline{i} = (i_n)_{n \geq 1} \in \Lambda^{\mathbb{N}}$ .

**2.2. Self-similar sets.** Let  $\mathcal{I}$  be an IFS as in (1.2) with non-empty compact attractor  $K \subset \mathbb{R}^d$  satisfying (1.3). For  $i = i_1 \dots i_n \in \Lambda^n$  write

$$f_i = f_{i_n} \circ \dots \circ f_{i_1} = r_i O_i \cdot + t_i,$$

where  $r_i = r_{i_n} \dots r_{i_1}$ ,  $O_i = O_{i_n} \dots O_{i_1}$  and  $t_i$  is the appropriate translation. Let  $\Phi : \Lambda^{\mathbb{N}} \mapsto K$  be the canonical projection, that is

$$\Phi(\underline{i}) = \lim_{n \rightarrow \infty} f_{\underline{i}|_n}(K).$$

Let  $R = \max\{|x| : x \in K\}$  and  $\rho = \max\{r_i : i \in \Lambda\}$ . Then it is easy to see that  $\Phi : (\Lambda^{\mathbb{N}}, d_\rho) \mapsto K$  is  $R$ -Lipschitz.

**2.3. Random multiplicative cascades.** A random multiplicative cascade is essentially a measure on  $\Lambda^{\mathbb{N}}$  constructed in a self-similar manner on the successive  $\Lambda^n$ , see [15, 2]. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let

$$W = (W_i)_{i \in \Lambda} \in [0, \infty)^m$$

be a random vector with  $\sum_{i \in \Lambda} \mathbb{E}(W_i) = 1$ . To avoid trivial cases, assume that

$$(a0) \quad \mathbb{P}(\#\{i \in \Lambda : W_i > 0\} > 1) > 0.$$

Let  $\{W^{[i]} : i \in \Lambda^*\}$  be a sequence of independent and identically distributed random vectors having the same law as  $W$ . For  $i \in \Lambda^*$ ,  $n \geq 1$  and  $j = j_1 \cdots j_n \in \Lambda^n$  define

$$Q_j^{[i]} = W_{j_1}^{[i]} W_{j_2}^{[ij_1]} \cdots W_{j_n}^{[ij_1 \cdots j_{n-1}]},$$

and for  $i \in \Lambda^*$  and  $n \geq 1$  define

$$Y_n^{[i]} = \sum_{j \in \Lambda^n} Q_j^{[i]}.$$

By definition  $\{Y_n^{[i]}\}_{n \geq 1}$  is a non-negative martingale. Assume also that

$$(a1) \quad \text{There exists } p > 1 \text{ such that } \sum_{i=1}^m \mathbb{E}(W_i^p) < 1.$$

Then  $Y_n^{[i]}$  converges almost surely to a nontrivial limit which we denote by  $Y^{[i]}$ , with expectation  $\mathbb{E}(Y^{[i]}) = 1$ . Moreover, for  $p > 1$  one has  $\mathbb{E}(Y^p) < \infty$  if and only if  $\sum_{i=1}^m \mathbb{E}(W_i^p) < 1$  (see [15, 5]). Since  $\Lambda^*$  is countable,  $Y^{[i]}$  is well-defined for all  $i \in \Lambda^*$  simultaneously. Moreover, by construction,

$$(2.1) \quad Y^{[i]} = \sum_{j=1}^m W_j^{[i]} Y^{[ij]}.$$

Then for each  $i \in \Lambda^*$  we may define a random measure  $\mu^{[i]}$  on  $\Lambda^{\mathbb{N}}$  by

$$(2.2) \quad \mu^{[i]}([j]) = Q_j^{[i]} \cdot Y^{[ij]}, \quad j \in \Lambda^*.$$

The measure  $\mu^{[i]}$  is called the *random multiplicative cascade measure* generated by the sequence  $\{W^{[ij]} : j \in \Lambda^*\}$ . By definition the sequence  $\{\mu^{[i]} : i \in \Lambda^*\}$  has the same law. Moreover, by (2.1) we have statistical self-similarity in the sense that for  $i \in \Lambda^*$  and  $j \in \Lambda^n$ ,

$$(2.3) \quad \mu^{[i]}|_{[j]} = Q_j^{[i]} \cdot \mu^{[ij]} \circ \sigma^{-n}|_{[j]}.$$

Sometimes we will write  $(\cdot) = (\cdot)^{[0]}$ . Our main interest will be in random cascade measures on the self-similar set  $K$  given by the canonical projection  $\Phi\mu^{[i]}$  of  $\mu^{[i]}$  onto  $K$ . For more on random cascade measures, see [3] and the references therein.

**2.4. The underlying probability space.** We now give a precise definition of the probability space on which the i.i.d. sequence  $\{W^{[i]} : i \in \Lambda^*\}$  is defined. First recall that the random vector  $W$  is defined on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will work on the countable product space

$$(\Omega^*, \mathcal{F}^*, \mathbb{P}^*) = \bigotimes_{i \in \Lambda^*} (\Omega_i, \mathcal{F}_i, \mathbb{P}_i),$$

where  $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) = (\Omega, \mathcal{F}, \mathbb{P})$  for each  $i \in \Lambda^*$ . For  $i \in \Lambda^*$  define the projection

$$\pi_i : \Omega^* \mapsto \Omega_i.$$

Then by letting  $W^{[i]} = W \circ \pi_i$  for  $i \in \Lambda^*$  we obtain a family of i.i.d. random vectors on  $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ .

For  $i \in \Lambda^*$  let  $\mu^{[i]} \equiv \mu^{[i]}(\cdot, \omega)$  be the random cascade measure generated by the sequence  $\{W^{[ij]} : j \in \Lambda^*\}$ , as in (2.2). For  $i \in \Lambda^*$  define

$$\eta_i : \Omega^* \ni (\omega_j)_{j \in \Lambda^*} \mapsto (\omega_{ij})_{j \in \Lambda^*} \in \Omega^*.$$

By definition  $W^{[ij]} = W^{[i]} \circ \eta_j$  for all  $i, j \in \Lambda^*$ , thus

$$(2.4) \quad \mu^{[ij]}(\cdot, \omega) = \mu^{[i]}(\cdot, \eta_j \omega).$$

Consequently, from (2.3), for any  $B \in \mathcal{B}$ ,

$$\begin{aligned} \mu^{[i]}(B \cap [j], \omega) &= Q_j^{[i]}(\omega) \cdot \mu^{[ij]}(\sigma^{-n}(B \cap [j]), \omega) \\ &= Q_j^{[i]}(\omega) \cdot \mu^{[i]}(\sigma^{-n}(B \cap [j]), \eta_j \omega). \end{aligned}$$

**2.5. The Peyrière measure.** Let  $(\Omega', \mathcal{F}') = (\Lambda^{\mathbb{N}} \times \Omega^*, \mathcal{B} \otimes \mathcal{F}^*)$ . Let  $\mathbb{Q}$  be the Peyrière measure on  $(\Omega', \mathcal{F}')$  with respect to  $\mu = \mu^{[0]}$  [15], that is for all  $A \in \mathcal{F}'$ ,

$$\mathbb{Q}(A) = \int_{\Omega^*} \int_{\Lambda^{\mathbb{N}}} \chi_A(\underline{i}, \omega) \mu(d\underline{i}, \omega) \mathbb{P}^*(d\omega).$$

It is easy to see that  $(\Omega', \mathcal{F}', \mathbb{Q})$  is a probability space. Define the skew product

$$T : \Omega' \ni (\underline{i}, \omega) \mapsto (\sigma \underline{i}, \eta_{\underline{i}|_1}(\omega)) \in \Omega'.$$

**Lemma 2.1.** *The Peyrière measure  $\mathbb{Q}$  is  $T$ -invariant.*

*Proof.* For all  $B \in \mathcal{F}'$  one has

$$\begin{aligned} \mathbb{Q}(T^{-1}B) &= \int_{\Omega^*} \int_{\Lambda^{\mathbb{N}}} \chi_{T^{-1}B}(\underline{i}, \omega) \mu(d\underline{i}, \omega) \mathbb{P}^*(d\omega) \\ &= \int_{\Omega^*} \int_{\Lambda^{\mathbb{N}}} \chi_B(\sigma \underline{i}, \eta_{\underline{i}|_1} \omega) \mu(d\underline{i}, \omega) \mathbb{P}^*(d\omega) \\ &= \sum_{j \in \Lambda} \int_{\Omega^*} \int_{[j]} \chi_B(\sigma \underline{i}, \eta_j \omega) \mu(d\underline{i}, \omega) \mathbb{P}^*(d\omega) \\ &= \sum_{j \in \Lambda} \int_{\Omega^*} W_j^{[0]}(\omega) \int_{[j]} \chi_B(\sigma \underline{i}, \eta_j \omega) \mu(d\sigma \underline{i}, \eta_j \omega) \mathbb{P}^*(d\omega) \\ &= \sum_{j \in \Lambda} \int_{\Omega^*} W_j^{[0]}(\omega) \int_{\Lambda^{\mathbb{N}}} \chi_B(\underline{i}, \eta_j \omega) \mu(d\underline{i}, \eta_j \omega) \mathbb{P}^*(d\omega) \\ &= \sum_{j \in \Lambda} \mathbb{E}(W_j) \mathbb{Q}(B) \\ &= \mathbb{Q}(B). \end{aligned}$$

This gives the conclusion.  $\square$

**2.6. Dimension and entropy.** Let  $f : X \mapsto Y$  be a continuous mapping between two metric spaces  $X$  and  $Y$ . For a Borel measure  $\mu$  on  $X$  we denote by

$$f\mu = \mu \circ f^{-1}$$

the pull-back measure of  $\mu$  on  $Y$  through  $f$ .

For a measure  $\mu$  and  $x \in \text{supp} \mu$  let

$$D_\mu(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}$$

whenever the limit exists. If for some  $\alpha \geq 0$  we have  $D_\mu(x) = \alpha$  for  $\mu$ -a.e.  $x$  we say that  $\mu$  is *exact dimensional*.

For  $0 < r < 1$  and  $\nu$  a probability measure supported by a compact subset  $A$  of  $\mathbb{R}^d$ , let

$$H_r(\nu) = - \int_A \log \nu(B(x, r)) \nu(dx).$$

The *lower entropy dimension* of  $\nu$  is defined as

$$\dim_e \nu = \liminf_{r \rightarrow 0} \frac{H_r(\nu)}{-\log r}$$

and the *Hausdorff dimension* of  $\nu$  is  $\dim_H \nu = \inf\{\dim_H A : \mu(A) > 0\}$ . Then

$$\dim_H \nu \leq \dim_e \nu,$$

with equality when  $\nu$  is exact dimensional, see [7, 8].

For any sub-Borel  $\sigma$ -algebra  $\mathcal{A}$  of  $\mathcal{B}$ , any finite partition  $\mathcal{P}$  of  $\Lambda^\mathbb{N}$ , and any probability measure  $\nu$  on  $\Lambda^\mathbb{N}$  we define the *conditional information*

$$\mathbf{I}_\nu(\mathcal{P} | \mathcal{A}) = - \sum_{B \in \mathcal{P}} \chi_B \log \mathbb{E}_\nu(\chi_B | \mathcal{A})$$

and the *conditional entropy*

$$\mathbf{H}_\nu(\mathcal{P} | \mathcal{A}) = \int_{\Lambda^\mathbb{N}} \mathbf{I}_\nu(\mathcal{P} | \mathcal{A})(\underline{i}) \nu(d\underline{i}).$$

For the trivial  $\sigma$ -algebra  $\mathcal{N} = \{\emptyset, \Lambda^\mathbb{N}\}$  we use the convention that  $\mathbf{I}_\nu(\mathcal{P}) = \mathbf{I}_\nu(\mathcal{P} | \mathcal{N})$  and  $\mathbf{H}_\nu(\mathcal{P}) = \mathbf{H}_\nu(\mathcal{P} | \mathcal{N})$ , see [10]

### 3. EXACT-DIMENSIONALITY AND DIMENSION CONSERVATION

In this section we establish the exact-dimensionality of random cascade measures on self-similar sets without any separation condition, as well as of the projections of the measures onto subspaces and of sliced measures. Dimension conservation for the measures then follows easily.

Let  $\pi \in \Pi_{d,k}$  be fixed. For  $\underline{i} \in \Lambda^\mathbb{N}$  define the fibre

$$[\underline{i}]_\pi = (\pi\Phi)^{-1}(\pi\Phi(\underline{i})).$$

Denote by  $\mathcal{P}_\pi = \{[\underline{i}]_\pi : \underline{i} \in \Lambda^\mathbb{N}\}$ . It is a measurable partition in the sense that the quotient space  $\Lambda^\mathbb{N}/\mathcal{P}_\pi$  is separated by a countable family of measurable sets, which can be taken as  $\{(\pi\Phi)^{-1}B_i\}$  where  $\{B_i\}$  is the sequence of closed cubes in  $\pi(\mathbb{R}^d)$  with rational vertices. Denote by  $\widehat{\mathcal{P}}_\pi$  the  $\sigma$ -algebra generated by  $\mathcal{P}_\pi$ . Due to the conditional measure theorem of Rohlin [28], given the measurable partition  $\mathcal{P}_\pi$ , for any probability measure  $\nu$  on  $(\Lambda^\mathbb{N}, \mathcal{B})$ , for every  $\underline{i}$  in a set of full  $\nu$ -measure, there is a probability measure  $\nu_{\underline{i}, \pi}$  defined on  $\mathcal{P}_\pi(\underline{i}) = [\underline{i}]_\pi$  such that for any measurable set  $B \in \mathcal{B}$ , the mapping  $\underline{i} \mapsto \nu_{\underline{i}, \pi}(B)$  is  $\widehat{\mathcal{P}}_\pi$ -measurable and

$$\nu(B) = \int_{\Lambda^\mathbb{N}} \nu_{\underline{i}, \pi}(B) \nu(d\underline{i}).$$

From the geometric point of view (see [23] for example), these conditional measures can be defined by ‘slicing’, in the sense that for  $\nu$ -a.e.  $\underline{i} \in \Lambda^\mathbb{N}$ , for  $\underline{i}' \in [\underline{i}]_\pi$  and

$r > 0$ , the  $\nu_{\underline{i}, \pi}$ -measure of the ball  $\Phi^{-1}B(\Phi(\underline{i}'), r)$  is given by the limit

$$\nu_{\underline{i}, \pi}(\Phi^{-1}B(\Phi(\underline{i}'), r)) = \lim_{\epsilon \rightarrow 0} \frac{\nu(\Phi^{-1}B(\Phi(\underline{i}'), r) \cap \Phi^{-1}\pi^{-1}(B(\pi\Phi(\underline{i}), \epsilon)))}{\nu(\Phi^{-1}\pi^{-1}(B(\pi\Phi(\underline{i}), \epsilon)))}.$$

For  $i \in \Lambda^*$  define the random probability measure

$$\bar{\mu}^{[i]} = \chi_{\{\|\mu^{[i]}\| > 0\}} \frac{\mu^{[i]}}{\|\mu^{[i]}\|},$$

with the convention that  $\bar{\mu} = \bar{\mu}^{[0]}$ . Write  $\mathbb{P}_*(A) = \mathbb{P}^*(A \cap \{\bar{\mu} \neq 0\})$  for  $A \in \mathcal{F}^*$  for the probability conditional on  $\bar{\mu}$  not vanishing.

For any continuous function  $f : \Lambda^{\mathbb{N}} \mapsto \mathbb{R}^d$  denote by  $\mathcal{B}_f$  the  $\sigma$ -algebra generated by  $f^{-1}\mathcal{B}(\mathbb{R}^d)$ .

For  $y \in \pi(K)$  write  $\bar{\mu}_{y, \pi} = \bar{\mu}_{\underline{i}, \pi}$  for any  $\underline{i} \in \Lambda^{\mathbb{N}}$  such that  $\pi\Phi(\underline{i}) = y$ .

Let  $\mathcal{P} = \{[i] : i \in \Lambda\}$ . Here is the main theorem of this section.

**Theorem 3.1.**  *$\mathbb{P}_*$ -almost surely*

(i)  $\Phi\mu$  is exact-dimensional with dimension

$$\alpha = \frac{\mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i}.$$

(ii)  $\pi\Phi\mu$  is exact-dimensional with dimension

$$\beta = \frac{\mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\pi\Phi})) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i}.$$

(iii) For  $\pi\Phi\mu$ -a.e.  $y \in \pi(K)$ ,  $\Phi\bar{\mu}_{y, \pi}$  is exact-dimensional with dimension

$$\gamma = \frac{\mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})) - \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\pi\Phi}))}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i}.$$

The dimension conservation result for measures now follows.

**Corollary 3.2.**  *$\mathbb{P}_*$ -almost surely for  $\pi\Phi\mu$ -a.e.  $y \in \pi(K)$ ,*

$$\dim_H \pi\Phi\bar{\mu} + \dim_H \Phi\bar{\mu}_{y, \pi} = \dim_H \Phi\bar{\mu}.$$

*Proof.* It follows from the fact that all these measures are exact-dimensional and  $\alpha - \beta = \gamma$ .  $\square$

**3.1. Proof of Theorem 3.1(i).** The proof is adapted from [10]. The difference is that here in the random setting we have used Komolgorov's zero-one law to deduce the ergodicity we need.

For  $n \geq 0$  and  $\underline{i} \in \Lambda^{\mathbb{N}}$  let

$$B_{\Phi}(\underline{i}, n) = \Phi^{-1}(B(\Phi(\underline{i}), r_{\underline{i}|_n})),$$

with the convention that  $r_{\emptyset} = 1$ . For  $\underline{i} \in \Lambda^{\mathbb{N}}$  denote by  $\mathcal{P}(\underline{i})$  the unique element of  $\mathcal{P}$  that contains  $\underline{i}$ . For  $n \geq 1$  let

$$f_n : \Lambda^{\mathbb{N}} \times \Omega_* \ni (\underline{i}, \omega) \mapsto \log \frac{\bar{\mu}(B_{\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_{\Phi}(\underline{i}, n))} \in \mathbb{R}.$$

From Lemma 3.3 and Proposition 3.5 in [10] we have that given any  $\omega \in \Omega^*$  such that  $\|\mu\| > 0$ , for  $\bar{\mu}$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$  we have

$$\lim_{n \rightarrow \infty} f_n(\underline{i}, \omega) = -\mathbf{I}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})(\underline{i}) := f(\underline{i}, \omega).$$

Furthermore, setting

$$\bar{f}(\underline{i}, \omega) = - \inf_{n \geq 1} \log \frac{\bar{\mu}(B_\Phi(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_\Phi(\underline{i}, n))},$$

there is a constant  $C_d$  (that depends only on  $d$ ) such that

$$\int_{\Lambda^\mathbb{N}} \bar{f}(\underline{i}, \omega) \bar{\mu}(d\underline{i}, \omega) \leq \mathbf{H}_{\bar{\mu}}(\mathcal{P}) + C_d.$$

This implies that  $\bar{f} \in L^1(\mathbb{Q})$ , provided  $\mathbb{E}(\|\mu\| \log \|\mu\|) < \infty$ .

Next we apply the following ergodic theorem due to Maker [18].

**Theorem 3.3. (Maker)** *Let  $(X, \mathcal{B}, \mu, T)$  be a measure-preserving system and let  $\{f_n\}$  be integrable functions on  $(X, \mathcal{B}, \mu)$ . If  $f_n(x) \rightarrow f(x)$  almost everywhere and if  $\sup_n |f_n(x)| = g(x)$  is integrable, then for almost every  $x$ ,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k}(T^k x) = f_\infty(x),$$

where  $f_\infty(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x)$ .

**Proposition 3.4.**  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^\mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k} \circ T^k(\underline{i}, \omega) = \int_{\Omega^*} \int_{\Lambda^\mathbb{N}} f(\underline{i}, \omega) \mathbb{Q}(d\underline{i}, d\omega).$$

*Proof.* Using Theorem 3.3, we have for  $\mathbb{Q}$ -a.e.  $(\underline{i}, \omega)$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k}(T^k(\underline{i}, \omega)) = f_\infty(\underline{i}, \omega),$$

where

$$f_\infty(\underline{i}, \omega) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(\underline{i}, \omega).$$

Note that for  $k \geq 1$ ,

$$f \circ T^k(\underline{i}, \omega) = -\mathbf{I}_{\bar{\mu}[\underline{i}]_k}(\mathcal{P} | \mathcal{B}_\Phi)(\sigma^k \underline{i}).$$

For  $n \geq 1$  define the random variable

$$(3.1) \quad A_n : \Omega' \ni (\underline{i}, \omega) \mapsto \sum_{j=j_1 \cdots j_n \in \Lambda^n} \chi_{\{\underline{i}|_n=j\}} W_{j_n}^{[j_1 \cdots j_{n-1}]}(\omega) \in [0, \infty).$$

A routine check shows that  $\{A_n\}_{n \geq 1}$  is an i.i.d. sequence. Notice that for  $k \geq 1$ ,  $f \circ T^k$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_k := \sigma\{A_{k+n} : n \geq 1\}$ . Thus from Komolgorov's zero-one law we get that  $\{f \circ T^k\}_{k \geq 1}$  is an ergodic sequence. This implies that for  $\mathbb{Q}$ -a.e.  $(\underline{i}, \omega) \in \Omega'$ ,

$$f_\infty(\underline{i}, \omega) = \int_{\Omega^*} \int_{\Lambda^\mathbb{N}} f(\underline{i}, \omega) \mathbb{Q}(d\underline{i}, d\omega),$$

hence the conclusion.  $\square$

The next lemma (an analogue of Lemma 5.3 in [10] for self-similar sets) relates the shift on symbolic space to its geometric effect on balls in  $\mathbb{R}^d$ .



**Lemma 3.5.** For  $\underline{i} \in \Lambda^{\mathbb{N}}$  and  $r > 0$  we have

$$\Phi^{-1}(B(\Phi(\underline{i}), r_{\underline{i}|_1} \cdot r)) \cap \mathcal{P}(\underline{i}) = \sigma^{-1} \Phi^{-1}(B(\Phi(\sigma \underline{i}), r)) \cap \mathcal{P}(\underline{i}).$$

*Proof.* For  $\underline{i} = i_1 i_2 \dots$  and  $r > 0$  we have

$$B(\Phi(\underline{i}), r_{\underline{i}|_1} \cdot r) = f_{i_1}(B(\Phi(\sigma \underline{i}), r)).$$

Thus

$$\Phi^{-1}(B(\Phi(\underline{i}), r_{\underline{i}|_1} \cdot r)) \cap \mathcal{P}(\underline{i}) = \Phi^{-1}(f_{i_1}(B(\Phi(\sigma \underline{i}), r))) \cap \mathcal{P}(\underline{i}).$$

As

$$\begin{aligned} \underline{j} &= j_1 j_2 \dots \in \Phi^{-1}(f_{i_1}(B(\Phi(\sigma \underline{i}), r))) \cap \mathcal{P}(\underline{i}) \\ \Leftrightarrow j_1 &= i_1, \Phi(\underline{j}) \in f_{i_1}(B(\Phi(\sigma \underline{i}), r)) \\ \Leftrightarrow j_1 &= i_1, f_{j_1}(\Phi(\sigma \underline{j})) \in f_{i_1}(B(\Phi(\sigma \underline{i}), r)) \\ \Leftrightarrow j_1 &= i_1, \Phi(\sigma \underline{j}) \in B(\Phi(\sigma \underline{i}), r) \\ \Leftrightarrow j_1 &= i_1, \underline{j} \in \sigma^{-1} \Phi^{-1}(B(\Phi(\sigma \underline{i}), r)) \\ \Leftrightarrow \underline{j} &\in \sigma^{-1} \Phi^{-1}(B(\Phi(\sigma \underline{i}), r)) \cap \mathcal{P}(\underline{i}) \end{aligned}$$

we get  $\Phi^{-1}(f_{i_1}(B(\Phi(\sigma \underline{i}), r))) \cap \mathcal{P}(\underline{i}) = \sigma^{-1} \Phi^{-1}(B(\Phi(\sigma \underline{i}), r)) \cap \mathcal{P}(\underline{i})$ , hence the conclusion.  $\square$

For  $\underline{i} \in \Lambda^{\mathbb{N}}$  and  $n \geq 1$ , conditioning on  $\mu([\underline{i}|_n]) > 0$ , one has

$$\begin{aligned} \frac{\mu(B_{\Phi}(\underline{i}, n))}{\mu^{[\underline{i}|_n]}(B_{\Phi}(\sigma^n \underline{i}, 0))} &= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\mu^{[\underline{i}|_{k+1}]}(B_{\Phi}(\sigma^{k+1} \underline{i}, n-k-1))} \\ &= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \cdot \frac{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))}{\mu^{[\underline{i}|_{k+1}]}(B_{\Phi}(\sigma^{k+1} \underline{i}, n-k-1))} \\ &= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \cdot \frac{\mu^{[\underline{i}|_k]}(\sigma^{-1} B_{\Phi}(\sigma^{k+1} \underline{i}, n-k-1) \cap \mathcal{P}(\sigma^k \underline{i}))}{\mu^{[\underline{i}|_{k+1}]}(B_{\Phi}(\sigma^{k+1} \underline{i}, n-k-1))} \\ &= \prod_{k=0}^{n-1} \frac{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\mu^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \cdot W_{i_{k+1}}^{[\underline{i}|_k]} \\ (3.2) \quad &= \prod_{k=0}^{n-1} \frac{\bar{\mu}^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\bar{\mu}^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \cdot W_{i_{k+1}}^{[\underline{i}|_k]}. \end{aligned}$$

From Proposition 3.4 one has that  $\mathbb{P}_*$ -almost surely for  $\bar{\mu}$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \frac{\bar{\mu}^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\bar{\mu}^{[\underline{i}|_k]}(B_{\Phi}(\sigma^k \underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \\ &= \int_{\Omega^*} \int_{\Lambda^{\mathbb{N}}} \mathbf{I}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})(\underline{i}) \mathbb{Q}(d\underline{i}, d\omega) \\ (3.3) \quad &= \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})). \end{aligned}$$

To complete the proof of (i) we need the following lemma.

**Lemma 3.6.**  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

- (1)  $\lim_{n \rightarrow \infty} \frac{1}{n} \log W_{i_{n+1}}^{[\underline{i}|n]} = \sum_{i=1}^m \mathbb{E}(W_i \log W_i)$ ;
- (2)  $\lim_{n \rightarrow \infty} \frac{1}{n} \log r_{\underline{i}|n} = \sum_{i=1}^m \mathbb{E}(W_i) \log r_i$ ;
- (3)  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|\mu^{[\underline{i}|n]}\| = 0$ ;
- (4)  $\lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{\mu}^{[\underline{i}|n]}(B_{\Phi}(\sigma^n \underline{i}, 0)) = 0$ .

*Proof.* (1) and (2) follow from the strong law of large numbers under the Peyrière measure  $\mathbb{Q}$ . (3) follows from [2, Theorem IV(ii)], provided that  $\mathbb{E}(\|\mu\|^{1+\epsilon}) < \infty$  for some  $\epsilon > 0$ .

To prove (4), first take  $a > 0$  and for  $n \geq 1$  define

$$E_n = \{\underline{i} \in \Lambda^{\mathbb{N}} : \bar{\mu}^{[\underline{i}|n]}(B_{\Phi}(\sigma^n \underline{i}, 0)) \leq e^{-na}\}.$$

Notice that  $B_{\Phi}(\underline{i}, 0) = \Phi^{-1}B(\Phi(\underline{i}), 1)$ , thus there is an integer  $k \geq 1$  such that  $[\underline{i}|k] \subset B_{\Phi}(\underline{i}, 0)$  holds for all  $\underline{i} \in \Lambda^{\mathbb{N}}$ . Hence

$$E_n \subset E'_n := \{\underline{i} \in \Lambda^{\mathbb{N}} : \bar{\mu}^{[\underline{i}|n]}([\sigma^n \underline{i}|_k]) \leq e^{-na}\}.$$

Then for any  $\eta \in (0, 1)$ ,

$$\begin{aligned} \mu(E'_n) &= \sum_{i \in \Lambda^{n+k}} \mu([i]) \cdot \chi_{\{\bar{\mu}^{[i|n]}([\sigma^n i]) \leq e^{-na}\}} \\ &\leq \sum_{i \in \Lambda^{n+k}} \mu([i]) \cdot \chi_{\{\mu([i]) > 0\}} (e^{-na} \cdot \bar{\mu}^{[i|n]}([\sigma^n i])^{-1})^{\eta} \\ &= e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot \mu([i]) \cdot \bar{\mu}^{[i|n]}([\sigma^n i])^{-\eta} \\ &= e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \mu^{[i|n]}([\sigma^n i])^{1-\eta} \cdot \|\mu^{[i|n]}\|^{-\eta} \\ &\leq e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \|\mu^{[i|n]}\|^{1-2\eta}. \end{aligned}$$

This implies that

$$\mathbb{E}(\mu(E'_n)) \leq e^{-a\eta n} \cdot m^k \cdot \mathbb{E}(\|\mu\|^{1-2\eta}).$$

Using the Borel-Cantelli lemma, and since  $a$  is arbitrary, it follows that  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \bar{\mu}^{[\underline{i}|n]}(B_{\Phi}(\sigma^n \underline{i}, 0)) \geq 0.$$

On the other hand since  $\|\bar{\mu}^{[i]}\| \leq 1$  for all  $i \in \Lambda^*$ , then  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \bar{\mu}^{[\underline{i}|n]}(B_{\Phi}(\sigma^n \underline{i}, 0)) \leq 0,$$

completing the proof of (4).  $\square$

Combining (3.2), (3.3) and Lemma 3.6 we have proved that  $\mathbb{P}_*$ -almost surely for  $\mu$ -almost every  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\lim_{n \rightarrow \infty} \frac{\log \Phi \mu(B(\Phi(\underline{i}), r_{\underline{i}|n}))}{\log r_{\underline{i}|n}} = \frac{\mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})) + \sum_{i=1}^m \mathbb{E}(W_i \log W_i)}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i},$$

which gives the conclusion.  $\square$

**3.2. Proof of Theorem 3.1(ii).** The proof is almost the same as that of (i), we can formally replace  $\Phi$  by  $\pi\Phi$ . The only difference is that Lemma 3.5 is replaced by the following analogue.

**Lemma 3.7.** *For  $\underline{i} \in \Lambda^{\mathbb{N}}$  and  $r > 0$  we have*

$$(\pi\Phi)^{-1}(B(\pi\Phi(\underline{i}), r_{\underline{i}1} \cdot r)) \cap \mathcal{P}(\underline{i}) = \sigma^{-1}(\pi\Phi)^{-1}(B(\pi\Phi(\sigma\underline{i}), r)) \cap \mathcal{P}(\underline{i}).$$

*Proof.* For  $\underline{i} = i_1 i_2 \dots$  and  $r > 0$  we have

$$B(\pi\Phi(\underline{i}), r_{\underline{i}1} \cdot r) = \pi f_{i_1}(B(\Phi(\sigma\underline{i}), r)).$$

Thus

$$(\pi\Phi)^{-1}(B(\pi\Phi(\underline{i}), r_{\underline{i}1} \cdot r)) \cap \mathcal{P}(\underline{i}) = \Phi^{-1}\pi^{-1}(\pi f_{i_1}(B(\Phi(\sigma\underline{i}), r))) \cap \mathcal{P}(\underline{i}).$$

But

$$\begin{aligned} \underline{j} &= j_1 j_2 \dots \in \Phi^{-1}\pi^{-1}(\pi f_{i_1}(B(\Phi(\sigma\underline{i}), r))) \cap \mathcal{P}(\underline{i}) \\ \Leftrightarrow j_1 &= i_1, \Phi(\underline{j}) \in \pi^{-1}(\pi f_{i_1}(B(\Phi(\sigma\underline{i}), r))) \\ \Leftrightarrow j_1 &= i_1, f_{j_1}(\Phi(\sigma\underline{j})) \in \pi^{-1}(\pi f_{i_1}(B(\Phi(\sigma\underline{i}), r))) \\ \Leftrightarrow j_1 &= i_1, \pi f_{j_1}(\Phi(\sigma\underline{j})) \in \pi f_{i_1}(B(\Phi(\sigma\underline{i}), r)) \\ \Leftrightarrow j_1 &= i_1, \pi\Phi(\sigma\underline{j}) \in \pi(B(\Phi(\sigma\underline{i}), r)) \\ \Leftrightarrow j_1 &= i_1, \pi\Phi(\sigma\underline{j}) \in B(\pi\Phi(\sigma\underline{i}), r) \\ \Leftrightarrow j_1 &= i_1, \underline{j} \in \sigma^{-1}(\pi\Phi)^{-1}(B(\pi\Phi(\sigma\underline{i}), r)) \\ \Leftrightarrow \underline{j} &\in \sigma^{-1}(\pi\Phi)^{-1}(B(\pi\Phi(\sigma\underline{i}), r)) \cap \mathcal{P}(\underline{i}), \end{aligned}$$

which gives the conclusion.  $\square$

**3.3. Proof of Theorem 3.1(iii).** Lemmas 3.5 and 3.7 give that for  $k \geq 1$ ,  $\mathbb{P}_*$  almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\begin{aligned} &\bar{\mu}_{\underline{i}, \pi}(B_{\Phi}(\underline{i}, k) \cap \mathcal{P}(\underline{i})) \\ &= \lim_{n \rightarrow \infty} \frac{\bar{\mu}(B_{\Phi}(\underline{i}, k) \cap B_{\pi\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n))} \\ &= \lim_{n \rightarrow \infty} \frac{\bar{\mu}(B_{\Phi}(\underline{i}, k) \cap B_{\pi\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))} \cdot \frac{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n))} \\ &= \lim_{n \rightarrow \infty} \frac{\bar{\mu}^{[\underline{i}1]}(B_{\Phi}(\sigma\underline{i}, k-1) \cap B_{\pi\Phi}(\sigma\underline{i}, n-1))}{\bar{\mu}^{[\underline{i}1]}(B_{\pi\Phi}(\sigma\underline{i}, n-1))} \cdot \frac{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}(B_{\pi\Phi}(\underline{i}, n))} \\ &= \bar{\mu}_{\sigma\underline{i}, \pi}^{[\underline{i}1]}(B_{\Phi}(\sigma\underline{i}, k-1)) \cdot \exp(-\mathbf{I}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\pi\Phi})(\underline{i})). \end{aligned}$$

This gives that

$$(3.4) \quad \frac{\bar{\mu}_{\underline{i}, \pi}(B_{\Phi}(\underline{i}, n))}{\bar{\mu}_{\sigma^n \underline{i}, \pi}^{[\underline{i}n]}(B_{\Phi}(\sigma^n \underline{i}, 0))} = \prod_{k=0}^{n-1} \frac{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}k]}(B_{\Phi}(\underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} \cdot e^{-\mathbf{I}_{\bar{\mu}}^{[\underline{i}k]}(\mathcal{P} | \mathcal{B}_{\pi\Phi})(\sigma^k \underline{i})}.$$

We need two further results:

**Proposition 3.8.**  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \frac{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}k]}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}k]}(B_{\Phi}(\underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} = \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi})).$$

*Proof.* For  $n \geq 1$  let

$$f_n(\underline{i}, \omega) = \log \frac{\bar{\mu}_{\underline{i}, \pi}(B_{\Phi}(\underline{i}, n) \cap \mathcal{P}(\underline{i}))}{\bar{\mu}_{\underline{i}, \pi}(B_{\Phi}(\underline{i}, n))}.$$

From [10, Lemma 3.3 & Prop. 3.5] we get that, for  $\mathbb{Q}$ -a.e.  $(\underline{i}, \omega) \in \Omega'$ , the sequence  $f_n$  converges to

$$f := -\mathbf{I}_{\bar{\mu}}(\mathcal{P} | \widehat{\mathcal{P}}_{\pi} \vee \mathcal{B}_{\Phi}) = -\mathbf{I}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\Phi}),$$

provided  $\mathbb{E}(\|\mu\| \log \|\mu\|) < \infty$ . Here we have used that the  $\sigma$ -algebra  $\widehat{\mathcal{P}}_{\pi}$  is a sub- $\sigma$ -algebra of  $\mathcal{B}_{\Phi}$ . As

$$\frac{1}{n} \log \prod_{k=0}^{n-1} \frac{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}]_k}(B_{\Phi}(\sigma^k \underline{i}, n-k))}{\bar{\mu}_{\sigma^k \underline{i}, \pi}^{[\underline{i}]_k}(B_{\Phi}(\underline{i}, n-k) \cap \mathcal{P}(\sigma^k \underline{i}))} = \frac{1}{n} \sum_{k=0}^{n-1} f_{n-k} \circ T^k(\underline{i}, \omega),$$

the conclusion follows as in the proof of Proposition 3.4.  $\square$

The next lemma is an analogue of Lemma 3.6(4) for slices of measures.

**Lemma 3.9.**  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^{\mathbb{N}}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{\mu}_{\sigma^n \underline{i}, \pi}^{[\underline{i}]_n}(B_{\Phi}(\sigma^n \underline{i}, 0)) = 0.$$

*Proof.* Take  $a > 0$  and for  $n \geq 1$  define

$$E_n = \{\underline{i} \in \Lambda^{\mathbb{N}} : \bar{\mu}_{\sigma^n \underline{i}, \pi}^{[\underline{i}]_n}(B_{\Phi}(\sigma^n \underline{i}, 0)) \leq e^{-an}\}.$$

Recall that there is  $k \geq 1$  such that  $[\underline{i}]_k \subset B_{\Phi}(\underline{i}, 0)$  holds for all  $\underline{i} \in \Lambda^{\mathbb{N}}$ . Thus

$$E_n \subset E'_n := \{\underline{i} \in \Lambda^{\mathbb{N}} : \bar{\mu}_{\sigma^n \underline{i}, \pi}^{[\underline{i}]_n}([\sigma^n \underline{i}]_k) \leq e^{-an}\}.$$

We shall use the fact that for any  $f \in L^1(\Lambda^{\mathbb{N}}, \mathcal{B}, \bar{\mu})$ ,

$$\int_{\Lambda^{\mathbb{N}}} f(\underline{i}) \bar{\mu}(d\underline{i}) = \int_{\Lambda^{\mathbb{N}}} \int_{\Lambda^{\mathbb{N}}} f(\underline{j}) \bar{\mu}_{\underline{i}, \pi}(d\underline{j}) \bar{\mu}(d\underline{i}).$$

This gives for each  $\eta > 0$ ,

$$\begin{aligned} \mu(E'_n) &= \sum_{\underline{i} \in \Lambda^{n+k}} \int_{[\underline{i}]} \chi_{\{\bar{\mu}_{\sigma^n \underline{i}, \pi}^{[\underline{i}]_n}([\sigma^n \underline{i}]) \leq e^{-na}\}} \mu(d\underline{i}) \\ &= \sum_{\underline{i} \in \Lambda^{n+k}} \chi_{\{\mu([\underline{i}]) > 0\}} \cdot Q_{\underline{i}|_n} \cdot \int_{[\sigma^n \underline{i}]} \chi_{\{\bar{\mu}_{\underline{i}, \pi}^{[\underline{i}]_n}([\sigma^n \underline{i}]) \leq e^{-an}\}} \mu^{[\underline{i}]_n}(d\underline{i}) \\ &= \sum_{\underline{i} \in \Lambda^{n+k}} \chi_{\{\mu([\underline{i}]) > 0\}} \cdot Q_{\underline{i}|_n} \cdot \|\mu^{[\underline{i}]_n}\| \\ &\quad \cdot \int_{\Lambda^{\mathbb{N}}} \int_{\Lambda^{\mathbb{N}}} \chi_{[\sigma^n \underline{i}]} \cdot \chi_{\{\bar{\mu}_{\underline{i}, \pi}^{[\underline{i}]_n}([\sigma^n \underline{i}]) \leq e^{-an}\}} \bar{\mu}_{\underline{i}, \pi}^{[\underline{i}]_n}(d\underline{j}) \bar{\mu}^{[\underline{i}]_n}(d\underline{i}) \\ &\leq \sum_{\underline{i} \in \Lambda^{n+k}} \chi_{\{\mu([\underline{i}]) > 0\}} \cdot Q_{\underline{i}|_n} \cdot \|\mu^{[\underline{i}]_n}\| \\ &\quad \cdot \int_{\Lambda^{\mathbb{N}}} \int_{\Lambda^{\mathbb{N}}} \chi_{[\sigma^n \underline{i}]} \cdot \bar{\mu}_{\underline{i}, \pi}^{[\underline{i}]_n}([\sigma^n \underline{i}])^{-\eta} e^{-a\eta n} \bar{\mu}_{\underline{i}, \pi}^{[\underline{i}]_n}(d\underline{j}) \bar{\mu}^{[\underline{i}]_n}(d\underline{i}). \end{aligned}$$

Notice that  $\bar{\mu}_{\underline{j},\pi}^{[i|n]}$  is carried by  $[\underline{j}]_\pi$ , thus  $\bar{\mu}_{\underline{j},\pi}^{[i|n]}([\sigma^n i]) = \bar{\mu}_{\underline{j},\pi}^{[i|n]}([\sigma^n i])$  is constant for  $\bar{\mu}_{\underline{j},\pi}^{[i|n]}$ -a.e.  $\underline{j} \in \Lambda^\mathbb{N}$ . This yields

$$\begin{aligned} \mu(E'_n) &\leq e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \|\mu^{[i|n]}\| \cdot \int_{\Lambda^\mathbb{N}} \bar{\mu}_{\underline{j},\pi}^{[i|n]}([\sigma^n i])^{1-\eta} \bar{\mu}^{[i|n]}(d\underline{j}) \\ &\leq e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \|\mu^{[i|n]}\| \cdot \left( \int_{\Lambda^\mathbb{N}} \bar{\mu}_{\underline{j},\pi}^{[i|n]}([\sigma^n i]) \bar{\mu}^{[i|n]}(d\underline{j}) \right)^{1-\eta} \\ &= e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \|\mu^{[i|n]}\| \cdot \bar{\mu}^{[i|n]}([\sigma^n i])^{1-\eta} \\ &\leq e^{-a\eta n} \sum_{i \in \Lambda^{n+k}} \chi_{\{\mu([i]) > 0\}} \cdot Q_{i|n} \cdot \|\mu^{[i|n]}\|. \end{aligned}$$

The rest of the proof is the same as that of Lemma 3.6(4).  $\square$

Combining (3.4), Proposition 3.8 and Lemma 3.9 we get that  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^\mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \bar{\mu}_{\underline{i},\pi}(B_\Phi(\underline{i}, n)) = \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_\Phi)) - \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\pi\Phi})),$$

which yields that  $\mathbb{P}_*$ -almost surely for  $\mu$ -a.e.  $\underline{i} \in \Lambda^\mathbb{N}$ ,

$$(3.5) \quad \lim_{r \rightarrow 0} \frac{\log \Phi \bar{\mu}_{\underline{i},\pi}(B(\Phi(\underline{i}), r))}{\log r} = \frac{\mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_\Phi)) - \mathbb{E}(\mathbf{H}_{\bar{\mu}}(\mathcal{P} | \mathcal{B}_{\pi\Phi}))}{\sum_{i=1}^m \mathbb{E}(W_i) \log r_i} = \gamma.$$

For  $y \in \pi\Phi(\Lambda^\mathbb{N})$  recall that  $\Phi \bar{\mu}_{y,\pi} = \Phi \bar{\mu}_{\underline{i},\pi}$  for any  $\underline{i} \in \Lambda^\mathbb{N}$  such that  $\pi\Phi(\underline{i}) = y$ . By definition for every Borel set  $A \subset \mathbb{R}^d$

$$\begin{aligned} \Phi \bar{\mu}(A) &= \int_{\Lambda^\mathbb{N}} \int_{\Phi([\underline{j}]_\pi)} \chi_A(z) \Phi \bar{\mu}_{\underline{j},\pi}(dz) \bar{\mu}(d\underline{j}) \\ &= \int_{y \in \pi(\mathbb{R}^d)} \int_{\pi^{-1}(y)} \chi_A(z) \Phi \bar{\mu}_{y,\pi}(dz) \pi \Phi \bar{\mu}(dy). \end{aligned}$$

This, together with (3.5), yields that  $\mathbb{P}_*$ -almost surely for  $\pi\Phi \bar{\mu}$ -a.e.  $y \in \pi(\mathbb{R}^d)$ , the measure  $\Phi \bar{\mu}_{y,\pi}$  is exact-dimensional with dimension  $\gamma$ , completing the proof of (iii).

#### 4. DIMENSION OF PROJECTIONS

In this section we generalize the results of [13] on projections and images under  $C^1$  functions without singularities to random cascade measures.

**4.1. A measure-valued ergodic sequence.** Let  $D = B(0, R)$  be the closed ball center 0 and radius  $R = \max\{|x| : x \in K\}$ . Denote by  $\mathcal{M}$  the family of probability measures on  $D$  and let  $\mathcal{B}_*$  be its weak- $\star$  topology.

Let  $G = \langle O_i : i \in \Lambda \rangle$  be the compact subgroup of  $O(d, \mathbb{R})$  generated by the orthogonal maps  $O_i$ ,  $i \in \Lambda$  and let  $\mathcal{B}_G$  be its Borel  $\sigma$ -algebra. Define the measurable map

$$\phi : \Omega' \ni (\underline{i}, \omega) \mapsto O_{\underline{i}|_1}^{-1} \in G.$$

Let  $T_\phi$  be the skew product of  $T$  and  $\phi$  on  $\Omega' \times G$ , that is

$$T_\phi : (\omega', g) \mapsto (T(\omega'), g\phi(\omega')).$$

With  $\xi$  as the normalised Haar measure on  $G$ , it follows easily from Lemma 2.1 that  $\mathbb{Q} \times \xi$  is  $T_\phi$ -invariant.

Define the measurable mapping

$$M : \Omega' \times G \ni (\underline{i}, \omega, g) \mapsto g\Phi\bar{\mu} \in \mathcal{M}.$$

Let  $M_0 = M$  and  $M_n = M \circ T_\phi^n$  for  $n \geq 1$ . By (2.4)

$$M_n(\underline{i}, \omega, g) = gO_{\underline{i}|n}^{-1}\Phi\bar{\mu}^{[\underline{i}|n]}.$$

Since  $\mathbb{Q} \times \xi$  is  $T_\phi$ -invariant,  $\overline{M} = (M_n)_{n \geq 0}$  forms a stationary sequence. In other words, if  $P$  is the distribution of  $\overline{M}$  on  $(\mathcal{M}^\mathbb{N}, \mathcal{B}_*^{\otimes \mathbb{N}})$ , that is  $P = \mathbb{Q} \times \xi \circ \overline{M}^{-1}$ , and  $S : (\nu_n)_{n \geq 0} \mapsto (\nu_{n+1})_{n \geq 0}$  is the left-shift mapping on  $\mathcal{M}^\mathbb{N}$ , then  $P$  is  $S$ -invariant. We use the group extension theorem for ergodic actions to show that it is also ergodic.

**Theorem 4.1.** *The dynamical system  $(\mathcal{M}^\mathbb{N}, \mathcal{B}_*^{\otimes \mathbb{N}}, P, S)$  is ergodic.*

*Proof.* For  $n \geq 1$  define

$$R_n : (\underline{i}, \omega, g) \mapsto \sum_{j=j_1 \dots j_n \in \Lambda^n} \chi_{\{\underline{i}|n=j\}} gO_{j_1}^{-1} \dots O_{j_n}^{-1}.$$

Also let  $R_0 : (\underline{i}, \omega, g) \mapsto g$ . Recall the i.i.d. sequence  $\{A_n\}_{n \geq 1}$  given by (3.1). We claim that  $\{A_n\}_{n \geq 1}$  and  $\{R_n\}_{n \geq 0}$  are independent. To see this, for any  $k \geq 1$  and  $l \geq 0$  and any bounded continuous functions  $h_1 \in C[0, \infty)$  and  $h_2 \in C(G)$ ,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q} \times \xi}(h_1(A_k)h_2(R_l)) &= \sum_{j \in \Lambda^{k+l}} \mathbb{E}_{\mathbb{P}^*} \left( \mu([j])h_1 \left( W_{j_k}^{[j|k-1]} \right) \int_G h_2(gO_{j_l}^{-1}) \xi(dg) \right) \\ &= \sum_{j \in \Lambda^{k+l}} \mathbb{E}_{\mathbb{P}^*} \left( \mu([j])h_1 \left( W_{j_k}^{[j|k-1]} \right) \right) \int_G h_2(g) \xi(dg) \\ &= \mathbb{E}_{\mathbb{Q} \times \xi}(h_1(A_k)) \mathbb{E}_{\mathbb{Q} \times \xi}(h_2(R_l)). \end{aligned}$$

Let  $\mathcal{F}_n^A = \sigma(A_{n+k} : k \geq 0)$  for  $n \geq 1$  and  $\mathcal{F}^R = \sigma(R_k : k \geq 0)$ , so that  $\mathcal{F}_1^A$  and  $\mathcal{F}^R$  are independent  $\sigma$ -fields. Observe that for  $n \geq 1$  the mapping  $M_n$  is  $\mathcal{F}_n^A \vee \mathcal{F}^R$ -measurable. Consequently, for any  $S$ -invariant set  $B \in \mathcal{B}_*^{\otimes \mathbb{N}}$ , the set  $B' = \overline{M}^{-1}B$  must belong to  $\mathcal{F}_\infty^A \vee \mathcal{F}^R$ , where  $\mathcal{F}_\infty^A = \bigcap_{n \geq 1} \mathcal{F}_n^A$ . Hence the conditional expectation  $\mathbb{E}(\chi_{B'} | \mathcal{F}_1^A)$  is independent of itself, and thus is almost surely constant, implying that  $B' \in \mathcal{F}^R$ .

The conclusion can now be deduced from the ergodicity of the dynamical system  $(\Omega' \times G, \mathcal{F}' \otimes \mathcal{B}_G, \mathbb{Q} \times \xi, T_\phi)$  conditioning on  $\mathcal{F}^R$ . This dynamical system is equivalent to

$$(\Lambda^\mathbb{N} \times G, \mathcal{B} \otimes \mathcal{B}_G, \mu_p \times \xi, \sigma_\phi),$$

where  $\mu_p$  is the Bernoulli measure on  $\Lambda^\mathbb{N}$  corresponding to the probability vector  $p = (\mathbb{E}(W_i))_{i \in \Lambda}$ , and  $\sigma_\phi$  is the group extension

$$\sigma_\phi(\underline{i}, g) = (\sigma \underline{i}, gO_{\underline{i}|1}^{-1}).$$

From the group extension theorem, see, for example, [26, Corollary 4.5], the dynamical system  $(\Lambda^\mathbb{N} \times G, \mathcal{B} \otimes \mathcal{B}_G, \mu_p \times \xi, \sigma_\phi)$ , as a compact group extension of the Bernoulli full-shift with  $\sigma_\phi$  having a dense orbit, is ergodic, giving the conclusion.  $\square$

We now use the separability of  $C(\mathcal{M})$  to get convergence of ergodic averages for all  $h \in C(\mathcal{M})$ .

**Corollary 4.2.**  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$  and  $\mu$ -almost every  $\underline{i}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} h(gO_{\underline{i}|n}^{-1} \Phi \bar{\mu}^{[\underline{i}|n]}) = \mathbb{E}_{\mathbb{Q} \times \xi}(h(g\Phi \bar{\mu}))$$

for all  $h \in C(\mathcal{M})$ .

*Proof.* Let  $\{h_k\}_{k \geq 1}$  be a countable dense sequence in  $C(\mathcal{M})$ . It follows from Theorem 4.1 that  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$  and  $\mu$ -almost every  $\underline{i}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} h_k(gO_{\underline{i}|n}^{-1} \Phi \bar{\mu}^{[\underline{i}|n]}) = \mathbb{E}_{\mathbb{Q} \times \xi}(h_k(g\Phi \bar{\mu})) \quad \text{for all } k \geq 1.$$

For any  $h \in C(\mathcal{M})$ , take a subsequence  $\{h'_k\}_{k \geq 1}$  of  $\{h_k\}_{k \geq 1}$  that converges to  $h$ . On the one hand, since  $\mathcal{M}$  is compact,  $h$  is bounded, so by the dominated convergence theorem

$$\lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q} \times \xi}(h'_k(g\Phi \bar{\mu})) = \mathbb{E}_{\mathbb{Q} \times \xi}(h(g\Phi \bar{\mu})).$$

On the other hand, for each  $N$ ,

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} h'_k(gO_{\underline{i}|n}^{-1} \Phi \bar{\mu}^{[\underline{i}|n]}) - \frac{1}{N} \sum_{n=0}^{N-1} h(gO_{\underline{i}|n}^{-1} \Phi \bar{\mu}^{[\underline{i}|n]}) \right| \leq \|h'_k - h\|_{\infty}.$$

Thus the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} h(gO_{\underline{i}|n}^{-1} \Phi \bar{\mu}^{[\underline{i}|n]})$$

exists and equals  $\lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q} \times \xi}(h'_k(g\Phi \bar{\mu})) = \mathbb{E}_{\mathbb{Q} \times \xi}(h(g\Phi \bar{\mu}))$ .  $\square$

**4.2. Lower bound for the dimension of projections.** We use the  $\rho$ -tree method in [13] to obtain close lower bounds for the dimensions of projections of measures. Let  $\rho = \max\{r_i : i \in \Lambda\}$  and  $c = \min\{r_i : i \in \Lambda\}$ . For  $i = i_1 \cdots i_n \in \Lambda^*$  write

$$r_i^- = r_{i_1} \cdots r_{i_{n-1}}.$$

For each  $q \geq 1$  we redefine the alphabet used for symbolic space to obtain one for which the contraction ratios do not vary too much:

$$\Lambda_q = \{i \in \Lambda^* : r_i^- > \rho^q \text{ and } r_i \leq \rho^q\}.$$

By definition

$$c\rho^q < r_i \leq \rho^q$$

for all  $i \in \Lambda_q$ . The canonical mapping  $\Phi_q : (\Lambda_q^{\mathbb{N}}, d_{\rho^q}) \mapsto K$  is  $R$ -Lipschitz where recall that  $R = \max\{|x| : x \in K\}$ . Setting

$$W_q^{[j]} = (Q_i^{[j]})_{i \in \Lambda_q}, \quad j \in \Lambda_q^*$$

gives a random cascade measure  $\mu_q$  on  $\Lambda_q^{\mathbb{N}}$ . Observe that it is the same random cascade measure as  $\mu$  on embedding  $\Lambda_q^{\mathbb{N}}$  into  $\Lambda^{\mathbb{N}}$ .

Let  $G_q = \overline{O_i : i \in \Lambda_q}$  and denote by  $\xi_q$  its normalised Haar measure. As before,  $\Pi_{d,k}$  is the set of orthogonal projections from  $\mathbb{R}^d$  onto its  $k$ -dimensional subspaces. For  $\pi \in \Pi_{d,k}$ ,  $q \in \mathbb{N}$  and  $\nu$  a measure on  $\mathbb{R}^d$ , define

$$e_q(\pi, \nu) = \frac{1}{q \log(1/\rho)} H_{\rho^q}(\pi\nu),$$

and

$$E_q(\pi) = \mathbb{E}_{\mathbb{P}^* \times \xi_q}(e_q(\pi, g\Phi\bar{\mu})).$$

Here is the basic lower bound for almost all projections of  $\bar{\mu}$  in terms of  $E_q(\pi)$ .

**Theorem 4.3.**  $\mathbb{P}_*$ -almost surely for  $\xi_q$ -almost every  $g$ ,

$$\dim_H(\pi g\Phi\bar{\mu}) \geq \frac{q \log(1/\rho)}{q \log(1/\rho) - \log c} E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi_{d,k}.$$

*Proof.* By Corollary 4.2, we have that  $\mathbb{P}_*$ -almost surely for  $\mu_q$ -almost every  $\underline{i}$  and  $\xi_q$ -almost every  $g$ ,

$$(4.1) \quad \frac{1}{N} \sum_{n=1}^N e_q(\pi, gO_{\underline{i}|n}^{-1} \Phi_q \bar{\mu}_q^{[\underline{i}|n]}) \rightarrow E_q(\pi) \text{ for all } \pi \in \Pi_{d,k}.$$

Using the strong law of large numbers it follows that  $\mathbb{P}_*$ -almost surely for  $\mu_q$ -almost every  $\underline{i} \in \Lambda_q^{\mathbb{N}}$ ,

$$\lim_{n \rightarrow \infty} \frac{\log Q_{\underline{i}|n}}{-n} = - \sum_{i \in \Lambda_q} \mathbb{E}(\chi_{\{W_{q,i} > 0\}} W_{q,i} \log W_{q,i}) \in (0, \infty),$$

so in particular,  $\mathbb{P}_*$ -almost surely for  $\mu_q$ -almost every  $\underline{i} \in \Lambda_q^{\mathbb{N}}$ ,

$$Q_{\underline{i}|n} > 0 \text{ for all } n \geq 1.$$

Identically,

$$\chi_{\{Q_{\underline{i}|n} > 0\}} \bar{\mu}_q^{[\underline{i}|n]} = \chi_{\{Q_{\underline{i}|n} > 0\}} \chi_{\{\|\mu_q^{[\underline{i}|n]}\| > 0\}} \cdot \frac{\mu_q^{[\underline{i}|n]}}{\|\mu_q^{[\underline{i}|n]}\|} = \sigma^n \bar{\mu}_{q, [\underline{i}|n]},$$

where

$$\bar{\mu}_{q, [\underline{i}|n]} = \chi_{\{\mu_q([\underline{i}|n]) > 0\}} \frac{\mu_q|_{[\underline{i}|n]}}{\mu_q([\underline{i}|n])},$$

so

$$\begin{aligned} H_{\rho^q}(\pi g O_{\underline{i}|n}^{-1} \Phi_q \chi_{\{Q_{\underline{i}|n} > 0\}} \bar{\mu}_q^{[\underline{i}|n]}) &= H_{\rho^q}(\pi g O_{\underline{i}|n}^{-1} \Phi_q \sigma^n \bar{\mu}_{q, [\underline{i}|n]}) \\ &= H_{\rho^q \cdot r_{\underline{i}|n}}(\pi g \Phi_q \bar{\mu}_{q, [\underline{i}|n]}) \\ &\leq H_{(c\rho^q)^{n+1}}(\pi g \Phi_q \bar{\mu}_{q, [\underline{i}|n]}). \end{aligned}$$

Hence, using (4.1),  $\mathbb{P}_*$ -almost surely for  $\mu_q$ -almost every  $\underline{i}$  and  $\xi_q$ -almost every  $g$ ,

$$\frac{1}{q \log(1/\rho)} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N H_{(c\rho^q)^{n+1}}(\pi g \Phi_q \bar{\mu}_{q, [\underline{i}|n]}) \geq E_q(\pi) \text{ for all } \pi \in \Pi_{d,k}.$$

The mapping  $f \equiv \pi g \Phi_q : ((\Lambda^q)^{\mathbb{N}}, d_{\rho^q}) \mapsto \mathbb{R}^k$  is  $R$ -Lipschitz. By [13, Theorem 5.4] there exist a  $\rho^q$ -tree  $X$  and maps  $(\Lambda^q)^{\mathbb{N}} \xrightarrow{h} X \xrightarrow{f'} \mathbb{R}^k$  such that  $f = f'h$ , where  $h$  is a tree morphism and  $f'$  is  $C$ -faithful (see [13, Definition 5.1]) for some constant  $C$  depending only on  $R$  and  $k$ . Then, applying [13, Proposition 5.3] for the  $c\rho^q$ -tree  $X$  (since the result is independent of the constant  $\rho$ ), one can find a constant  $C'$  depending only on  $C$  and  $k$  such that for all  $n \geq 1$ ,

$$|H_{(c\rho^q)^{n+1}}(f \bar{\mu}_{q, [\underline{i}|n]}) - H_{(c\rho^q)^{n+1}}(h \bar{\mu}_{q, [\underline{i}|n]})| \leq C'.$$



Consequently,  $\mathbb{P}_*$ -almost surely for  $\bar{\mu}_q$ -almost every  $\underline{i}$  and  $\xi_q$ -almost every  $g$ ,

$$\frac{1}{q \log(1/\rho)} \liminf_{N \rightarrow \infty} \frac{1}{N-1} \sum_{n=1}^N H_{(c\rho^q)^{n+1}}(h\bar{\mu}_{q, [\underline{i}|n]}) \geq E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi_{d,k}.$$

By [13, Theorem 4.4] it follows that  $\mathbb{P}_*$ -almost surely for  $\xi_q$ -almost every  $g$ ,

$$\dim_H h\bar{\mu}_q \geq \frac{q \log(1/\rho)}{q \log(1/\rho) - \log c} E_q(\pi) - O(1/q) \text{ for all } \pi \in \Pi_{d,k}.$$

Since  $f'$  is  $C$ -faithful and  $f'h\bar{\mu}_q = f\bar{\mu}_q = \pi g\Phi_q\bar{\mu}_q = \pi g\Phi\bar{\mu}$ , the conclusion follows from [13, Proposition 5.2].  $\square$

**4.3. Projection theorems.** The projection results in [13] required the strong separation condition on the underlying IFS  $\mathcal{I}$ . With the approach of Sections 4.1 and 4.1 we avoid the need for any separation condition at all. Moreover, our results apply to random cascade measures as well as deterministic measures on self-similar sets.

To show that the dimension of projections is constant for all projections we need to assume that the group  $\langle O_i : i \in \Lambda \rangle$  is dense in  $SO(d, \mathbb{R})$ ; when this is the case we say that the IFS  $\mathcal{I}$  has *dense rotations*. Under this assumption, for each  $q \geq 1$ ,  $G_q = SO(d, \mathbb{R})$  so the normalised Haar measures  $\xi_q$  on  $SO(d, \mathbb{R})$  are the same and we write  $\xi$  for this measure.

**Theorem 4.4.** *Let  $\mathcal{I}$  have dense rotations. Then the limit*

$$E(\pi) := \lim_{q \rightarrow \infty} E_q(\pi)$$

*exists for every  $\pi \in \Pi_{d,k}$ , and  $E : \Pi_{d,k} \mapsto [0, k]$  is lower semi-continuous. Moreover:*

- (i)  $E(\pi) = \min(k, \alpha)$  for almost every  $\pi \in \Pi_{d,k}$ .
- (ii) For a fixed  $\pi \in \Pi_{d,k}$ ,  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$ ,

$$\dim_e \pi g\Phi\bar{\mu} = \dim_H \pi g\Phi\bar{\mu} = E(\pi).$$

(Recall that  $\dim_e$  is the entropy dimension.)

- (iii)  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$ ,

$$\dim_H \pi g\Phi\bar{\mu} \geq E(\pi) \text{ for all } \pi \in \Pi_{d,k}.$$

*Proof.* Using Theorem 3.1(1) and Theorem 4.3, the proof is similar to that of [13, Theorem 8.2].  $\square$

We can now conclude that the dimension of the projected measure is constant over all projections.

**Corollary 4.5.** *If  $\mathcal{I}$  has dense rotations then  $\mathbb{P}_*$ -almost surely*

$$\dim_H \pi\Phi\mu = \min(k, \alpha) \text{ for all } \pi \in \Pi_{d,k}.$$

*Proof.* Since  $E$  is lower semi-continuous, it follows from Theorem 4.4(i) that for any  $\epsilon > 0$  the set

$$\mathcal{U}_\epsilon = \{\pi \in \Pi_{d,k} : E(\pi) > \min(k, \alpha) - \epsilon\}$$

is open and dense in  $\Pi_{d,k}$ . Write  $\mathcal{U}_\epsilon \cdot g = \{\pi g : \pi \in \mathcal{U}_\epsilon\}$  for  $g \in SO(d, \mathbb{R})$ . Then from Theorem 4.4 (iii) one has  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$ ,

$$\tilde{\mathcal{U}}_\epsilon = \{\pi \in \Pi_{d,k} : \dim_H \pi\Phi\bar{\mu} > \min(k, \alpha) - \epsilon\} \supseteq \mathcal{U}_\epsilon \cdot g.$$

Since  $\mathcal{U}_\epsilon$  has non-empty interior, we deduce that  $\mathbb{P}_*$ -almost surely  $\tilde{\mathcal{U}}_\epsilon = \Pi_{d,k}$ , as required.  $\square$

As in [13] projection may be generalized to  $C^1$ -maps without singular points, that is  $C^1$ -maps for which the derivative matrix is everywhere non-singular.

**Proposition 4.6.** *If  $\mathcal{I}$  has dense rotations then for all  $C^1$ -maps  $h : B(0, R) \mapsto \mathbb{R}^k$  such that  $\sup_{x \in K} \|D_x h - \pi\| < c\rho^q$ , we have that  $\mathbb{P}_*$ -almost surely for  $\xi$ -almost every  $g$ ,*

$$\dim_H hg\Phi\bar{\mu} \geq E_q(\pi) - O(1/q).$$

*Proof.* The proof is similar to that of [13, Proposition 8.4].  $\square$

**Corollary 4.7.** *If  $\mathcal{I}$  has dense rotations then  $\mathbb{P}_*$ -almost surely, for all  $C^1$ -maps  $h : K \mapsto \mathbb{R}^k$  without singular points,*

$$\dim_H h\Phi\mu = \min(k, \alpha).$$

*Proof.* This follows from Proposition 4.6 and Corollary 4.5.  $\square$

## 5. APPLICATIONS TO SELF-SIMILAR SETS

Random cascade measures include non-random measures as a special case, and these we apply to the fractal geometry of deterministic self-similar sets. As before, we consider an IFS of similarities

$$(5.1) \quad \mathcal{I} = \{f_i = r_i O_i \cdot + t_i\}_{i=1}^m$$

where  $0 < r_i < 1$  are the scaling ratios,  $O_i$  are orthonormal transformations, and  $t_i$  are translations. We write  $K$  for the self-similar set that is the attractor of the IFS, that is the unique non-empty compact set satisfying  $K = \cup_{i=1}^m f_i(K)$ . Recall that the IFS  $\mathcal{I}$  satisfies the *strong separation condition* (SSC) if this union is disjoint and the *open set condition* (OSC) if there is a non-empty open set  $V$  such that  $V \subseteq \cup_{i=1}^m f_i(V)$  with this union disjoint. If either SSC or OSC are satisfied then

$$(5.2) \quad \dim_H K = s \quad \text{where} \quad \sum_{i=1}^m r_i^s = 1.$$

To enable us to transfer the results to sets directly we need to ensure that the sets support suitable measures. We say that a self similar set satisfies the *strong variational principle* if there is a Bernoulli probability measure  $\mu$  on  $\Lambda^\mathbb{N}$  such that  $\dim_H \Phi\mu = \dim_H K$ . The strong variational principle holds in many cases.

**Lemma 5.1.** (a) *Let  $K$  be the self-similar attractor of an IFS  $\mathcal{I}$  satisfying the open set (or strong separation) condition. Then  $K$  satisfies the strong variational principle.*

(b) *Given  $0 < r_i < \frac{1}{2}$  and  $O_i$ , the IFS  $\mathcal{I}$  in (5.1) satisfies the strong variational principle for almost all  $(t_1, \dots, t_m)$  in the sense of  $md$ -dimensional Lebesgue measure.*

*Proof.* (a) With  $s$  given by (5.2), the Bernoulli probability measure  $\mu$  on  $\Lambda^\mathbb{N}$ , defined by

$$(5.3) \quad \mu^{[0]}([i]) = r_i^s \quad (i = 1, \dots, m),$$

has  $\dim_H \Phi\mu = \dim_H K$ . This fact is the key step in showing that  $\dim_H K = s$  when OSC holds, see for example [14].

(b) This follows by applying to self-similar sets the argument used in [6] to find the almost sure dimension of self-affine sets. With  $\mu$  as in (5.3), integrating the  $t$ -energy of the image measures  $\Phi\mu$  over a parameterized family of self-similar sets gives that the energy is bounded for almost all parameters  $(t_1, \dots, t_m)$  if  $t < s$ , in which case  $\dim_H K = s$ .  $\square$

Furstenberg [11] introduced the notion of dimension conservation: given  $K \subseteq \mathbb{R}^d$  a projection,  $\pi \in \Pi_{d,k}$  is said to be *dimension conserving* if there is a number  $\Delta > 0$  such that

$$(5.4) \quad \Delta + \dim_H \{y \in \mathbb{R}^k : \dim_H(K \cap \pi^{-1}y) \geq \Delta\} \geq \dim_H K$$

(here we take  $\dim \emptyset = -\infty$ ). The reverse inequality to (5.4) holds for all  $0 \leq \Delta \leq d - k$  for all sets  $K$ , see [21]. Furstenberg [11] showed that self-similar attractors of IFSs with no rotational component, as well as certain other ‘homogeneous’ sets, are dimension conserving. (Note that, as is the case for Sierpiński carpets, the value of  $\Delta$  satisfying (5.4) may vary with the projection  $\pi$ .) The next Corollary extends this to all self-similar sets satisfying the strong variational principle.

**Corollary 5.2.** *Let  $K$  be the self-similar attractor of an IFS  $\mathcal{I}$  on  $\mathbb{R}^d$  satisfying the strong variational principle. Then all projections  $\pi \in \Pi_{d,k}$  are dimension conserving, i.e. (5.4) holds for some  $0 \leq \Delta \leq d - k$ . In particular this is true if  $\mathcal{I}$  satisfies OSC.*

*Proof.* Let  $\mu$  be the Bernoulli measure on  $\Lambda^{\mathbb{N}}$  guaranteed by Lemma 5.1 such that  $\dim_H \Phi\mu = \dim_H K$ , so  $\pi\Phi\mu$  is supported by  $\pi K$ . By Corollary 3.2, for each  $\pi \in \Pi_{d,k}$ , we have for  $\pi\Phi\mu$ -almost all  $y \in \pi K$ ,

$$\dim_H \pi\Phi\mu + \dim_H \Phi\mu_{y,\pi} = \dim_H \Phi\mu.$$

Then, for  $\pi\Phi\mu$ -almost all  $y \in \pi K$ ,

$$(5.5) \quad \dim_H \pi K + \dim_H(K \cap \pi^{-1}y) \geq \dim_H \Phi\mu = \dim_H K,$$

which is dimension conservation taking  $\Delta = \dim_H K - \dim_H \pi K$ .  $\square$

If the IFS  $\mathcal{I}$  does not satisfy the strong variational principle, a weaker conclusion is still possible. Orponen used a covering argument [25, Lemma 3.4] (presented in  $\mathbb{R}^2$  but equally valid in  $\mathbb{R}^d$ ) to show that for all  $\epsilon > 0$  there is an IFS of similarities  $\mathcal{I}_\epsilon$  satisfying the strong separation condition, with attractor  $K_\epsilon \subseteq K$  such that  $\dim_H K_\epsilon > \dim_H K - \epsilon$ . Applying Corollary 5.2 to  $K_\epsilon$  then gives (5.4) but with  $\dim_H K - \epsilon$  on the right-hand side of the inequality.

The next two Corollaries weaken the conditions that ensure constant dimension of projections and images from those given in [13] to just requiring the strong variational principle. Recall that an IFS on  $\mathbb{R}^d$  of the form (5.1) has *dense rotations* if the group  $\langle O_i : i \in \Lambda \rangle$  is dense in  $SO(d, \mathbb{R})$ .

**Corollary 5.3.** *Let  $K$  be the self-similar attractor of an IFS  $\mathcal{I}$  which satisfies the strong variational principle and has dense rotations. Then*

$$\dim_H \pi K = \min(k, \dim_H K) \text{ for all } \pi \in \Pi_{d,k}.$$

*In particular this is true if the IFS  $\mathcal{I}$  satisfies OSC.*

*Proof.* Applying Lemma 5.1 and Corollary 4.5 in the deterministic setting with  $\mu$  defined by (5.3) gives, for all  $\pi \in \Pi_{d,k}$ ,

$$\min(k, \dim_H \Phi\mu) = \dim_H \pi\Phi\mu \leq \dim_H \pi K$$

since  $\pi K$  supports  $\pi\Phi\mu$ . The opposite inequality follows since  $\pi$  is a Lipschitz mapping.  $\square$

**Corollary 5.4.** *Let  $K$  be the self-similar attractor of an IFS  $\mathcal{I}$  which satisfies the strong variational principle and has dense rotations. Then for all  $C^1$ -maps  $g : K \mapsto \mathbb{R}^k$  without singular points,*

$$\dim_H g(K) = \min(k, \dim_H K).$$

*Proof.* This is similar to Corollary 5.3 but using Corollary 4.7.  $\square$

Recall that for  $A \subseteq \mathbb{R}^d$  the *distance set* of  $A$  is defined as  $D(A) = \{|x - y| : x, y \in A\}$  and the *pinned distance set* of  $A$  at  $a$  is  $D_a(A) = \{|x - a| : x \in A\}$ . A general open problem is to relate the Hausdorff dimensions of  $D(A)$  and  $D_a(A)$  to that of  $A$ . For self-similar sets in the plane [25] showed that if  $\dim_H K > 1$  then  $\dim_H D(K) \geq 1$ . We have the following variant.

**Corollary 5.5.** *Let  $K$  be the self-similar attractor of an IFS  $\mathcal{I}$  which satisfies the strong variational principle and has dense rotations. Then there exists  $a \in K$  such that*

$$\dim_H D_a(K) = \min(k, \dim_H K),$$

*so in particular*

$$\dim_H D(K) \geq \min(k, \dim_H K).$$

*Proof.* Take a point  $a \in K$ , and some  $i \in \Lambda^*$  such that  $a \notin f_i(K)$ . Then  $f_i(K)$  is the self-similar attractor of the IFS  $\{f_{i1}, \dots, f_{im}\}$  which satisfies the variational principle and has dense rotations. The mapping  $h : f_i(K) \rightarrow \mathbb{R}$  by  $h(x) = |x - a|$  is  $C^1$  and has no singular points, so applying Corollary 4.7 to  $f_i(K)$  gives

$$\dim_H h\{|x - a| : x \in f_i(K)\} = \min(k, \dim_H f_i(K)) = \min(k, \dim_H K)$$

since  $f_i(K)$  is similar to  $K$ . Since  $a \in K$  and  $f_i(K) \subseteq K$ ,  $h\{|x - a| : x \in f_i(K)\} \subset D_a(K)$  and the conclusions follow.  $\square$

## 6. THE PERCOLATION MODEL

Whilst fractal percolation or Mandelbrot percolation is most often based on a decomposition of a  $d$ -dimensional cube into  $m^d$  equal subcubes of sides  $m^{-1}$ , random subsets of any self-similar set may be constructed using a similar percolation process. Let  $\mathcal{I} = \{f_i = r_i O_i \cdot + t_i\}_{i=1}^m$  be an IFS of similarities. Let  $K$  be its attractor and let  $\mathbb{P}$  be a probability distribution on the  $\mathbf{P}(\Lambda)$ , the collection of all subsets of  $\Lambda = \{1, \dots, m\}$ . We define a sequence of random subset of  $\Lambda^n$  inductively as follows. The random set  $S_1 \subseteq \Lambda$  has distribution  $\mathbb{P}$ . Then, given  $S_n$ , let  $S_{n+1} = \cup_{i \in S_n} S^i$  where  $S^i = \{ij : j \in S_1^i\} \subseteq \Lambda^{n+1}$  and where  $S_1^i \subseteq \Lambda$  has the distribution  $\mathbb{P}$  independently for each  $i \in S_n$ . A sequence of random subsets  $\{K_n\}_{n=1}^\infty$  of  $K$  is given by  $K_n = \cup_{i \in S_n} f_i(K)$ . We write  $K_\mathbb{P} = \cap_{n=0}^\infty K_n$  for the resulting random compact subset of  $K$  which is known as the *percolation set*.

We say that  $(\mathcal{I}, \mathbb{P})$  satisfies the *strong variational principle* if there exists a random cascade measure  $\mu$  such that, conditional on  $K_\mathbb{P} \neq \emptyset$ ,

$$(6.1) \quad \dim_H K_\mathbb{P} = \dim_H \Phi\mu = \alpha,$$

where  $\alpha$  is given by Theorem 3.1 (i).

**Lemma 6.1.** *If  $\mathcal{I}$  satisfies OSC and if  $\mathbb{E}\{\text{card}S_1\} > 1$ , then  $(\mathcal{I}, \mathbb{P})$  satisfies the strong variational principle, with  $\alpha$  given by  $\mathbb{E}(\sum_{i \in S_1} r_i^\alpha) = 1$ .*

*Proof.* By standard branching process theory [1], if  $\mathbb{E}\{\text{card}S_1\} > 1$  there is a positive probability that  $K_{\mathbb{P}} \neq \emptyset$ . Under OSC, conditional on  $K_{\mathbb{P}} \neq \emptyset$ , the dimension of  $K_{\mathbb{P}}$  is given by the solution  $\alpha$  of  $\mathbb{E}(\sum_{i \in S_1} r_i^\alpha) = 1$ , see [6, 24]. We may associate a random cascade with this percolation process by defining the random vector  $W$  as

$$(6.2) \quad W = (W_1, \dots, W_n) = (r_1^s \chi_{\{1 \in S\}}(\omega), \dots, r_m^s \chi_{\{m \in S\}}(\omega)).$$

This defines a random cascade measure  $\Phi\mu$  supported by  $K_{\mathbb{P}}$ . Moreover, using a potential-theoretic estimate (or a direct verification of the formula in Theorem 3.1 (i) that  $\alpha = s$ ),  $\dim_H \Phi\mu = \dim_H K_{\mathbb{P}}$  almost surely, see [6, 24].  $\square$

Our first application is the following dimension conservation result for percolation sets.

**Corollary 6.2.** *Suppose that  $(\mathcal{I}, \mathbb{P})$  satisfies the strong variational principle. Then for any projection  $\pi \in \Pi_{d,k}$  almost surely conditional on  $K_{\mathbb{P}} \neq \emptyset$ , one has*

$$\Delta + \dim_H \{y \in \mathbb{R}^k : \dim_H(K_{\mathbb{P}} \cap \pi^{-1}y) \geq \Delta\} \geq \dim_H K_{\mathbb{P}}$$

for some  $0 \leq \Delta \leq d - k$ .

*Proof.* It follows the same lines as in the proof of Corollary 5.2.  $\square$

Investigation of the dimensions of projections of the basic  $m$ -adic square-based percolation process goes back some years, see [4] for a survey, and recently Rams and Simon [29] showed using direct geometric arguments that almost surely all orthogonal projections of square-based percolation have Hausdorff dimension  $\min\{1, \alpha\}$ , where  $\alpha$  is the dimension of the percolation set. The following corollary gives a similar conclusion for percolation on self-similar sets for which the IFS has dense rotations.

**Corollary 6.3.** *Suppose that  $(\mathcal{I}, \mathbb{P})$  satisfies the strong variational principle and has dense rotations. Then almost surely*

$$\dim_H \pi K_{\mathbb{P}} = \min(k, \alpha) \text{ for all } \pi \in \Pi_{d,k},$$

conditional on  $K_{\mathbb{P}} \neq \emptyset$ , where  $\alpha$  is given by (6.1).

*Proof.* This follows by applying Corollary 4.6 to the random cascade measure  $\mu$  given in (6.1).  $\square$

Again there is a variation for  $C^1$ -maps without singular points.

**Corollary 6.4.** *Suppose that  $(\mathcal{I}, \mathbb{P})$  satisfies the strong variational principle and has dense rotations and. Then almost surely for all  $C^1$ -maps  $g : K \rightarrow \mathbb{R}^k$  without singular points,*

$$\dim_H g(K_{\mathbb{P}}) = \min(k, \alpha),$$

conditional on  $K_{\mathbb{P}} \neq \emptyset$ , where  $\alpha$  is given by (6.1).

*Proof.* This follows by applying Corollary 4.7 to  $\mu$ .  $\square$

Distance sets of percolation sets have also attracted interest recently, see [29] for the case of square-based percolation. Here we address this problem for percolation on general self-similar sets with dense rotations.

**Corollary 6.5.** *Suppose that  $(\mathcal{I}, \mathbb{P})$  satisfies the strong variational principle and has dense rotations. Then almost surely, conditional on  $K_{\mathbb{P}} \neq \emptyset$ , there exists  $a \in K_{\mathbb{P}}$  such that*

$$\dim_H D_a(K_{\mathbb{P}}) = \min(1, \alpha),$$

*so in particular*

$$\dim_H D(K_{\mathbb{P}}) \geq \min(1, \alpha),$$

*where  $\alpha$  is given by (6.1).*

*Proof.* By taking two sub-processes of the percolation process,  $K_{\mathbb{P}}^1 \subseteq f_1(K)$  and  $K_{\mathbb{P}}^2 \subseteq f_2(K)$ , say, we may fix a point  $a \in K_{\mathbb{P}}^1 \setminus K_{\mathbb{P}}^2$  subject to non-extinction of  $K_{\mathbb{P}}^1$ , and then define  $h : \mathbb{R}^d \rightarrow \mathbb{R}$  by  $h(x) = |x - a|$ . The mapping  $h$  is  $C^1$  and has no singular points outside  $K_{\mathbb{P}}^1$ . Subject to non-extinction of  $K_{\mathbb{P}}^2$ , the set  $D_a(K_{\mathbb{P}}^2) = h(K_{\mathbb{P}}^2)$  almost surely has Hausdorff dimension  $\min(1, \alpha)$  by Corollary 6.4. A similar argument is valid within any component of the construction of  $K_{\mathbb{P}}$  so the conclusion holds almost surely, conditional on  $K_{\mathbb{P}} \neq \emptyset$ .  $\square$

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